Notes on Tate's Thesis

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Time: Fall 2015 (?).

Tate's Thesis §1: Abstract Fourier Transform

Let G be a locally compact abelian gp (LCAG).

e.g. $S^1 \subseteq \mathbb{C}^\times$, local fields, Adeles, Ideles.

Thm* G is a LCAG, then $\exists$ regular borel measure μ, s.t. $\forall x \in G$, $E \subseteq G$ measurable, $\mu(x+E) = \mu(E)$. Call μ a Haar measure. Moreover, it's unique up to scaling.

e.g. $\mathbb{R}$, μ = Lebesgue, $\mathbb{C}$, μ = 2 Lebesgue, K, $\mu(\mathcal{O}_K) = 1 \rightsquigarrow \mu(\mathbb{A}_K/K) = 1$

Defn. Let G be a LCAG, $\widehat{G} = \{\chi : G \to S^1 \mid \chi \text{ cont. hom.}\}$ w/ cpt-open top.

Thm* $\widehat{G}$ is again LCAG, and $\widehat{\widehat{G}} \cong G$ via the canonical embedding

e.g. 1. $\mathbb{R}$, $\widehat{\mathbb{R}} \cong \mathbb{R}$ via $y \longmapsto (x \longmapsto e^{2\pi i x y})$.

2. $G = \mathbb{Z}$, $\widehat{\mathbb{Z}} = \{\text{Hom from } \mathbb{Z} \text{ to } S^1\} \simeq S^1$

3. $\psi : \mathbb{Q}_p \longrightarrow S^1$ be $\psi(\sum_{i=-n}^{\infty} a_i p^i) = e^{2\pi i \sum_{i=-n}^{-1} a_i p^i}$

turns out $y \longmapsto \psi_y(x) = \psi(xy)$ is an iso. $\mathbb{Q}_p \xrightarrow{\sim} \widehat{\mathbb{Q}_p}$.

4. L/K finite extn, $K = \mathbb{R}, \mathbb{Q}_p$, ψ be standard char. in 1&3.

Define $\psi_L = \psi_K \circ \mathrm{tr}_{L/K}$, $y \longmapsto \psi_{L,y}(x) = \psi_L(xy)$ is an iso.: $L \longrightarrow \widehat{L}$.

Defn. Let G be a LCAG w/ Haar measure dx, then define for $f \in L^1(G)$ $\hat{f}(\psi) = \int_G f(x)\psi(x)\,dx$. So $\hat{\psi}$ is a function on $\widehat{G}$.

Thm* In the situation as above, $\exists$ Haar measure on $\hat{G}$, call it $d\psi$, s.t. $\forall f \in L^1(G)$, w/ $\hat{f} \in L^1(\hat{G})$, we have,

$$f(x) = \int_{\hat{G}} \hat{f}(\psi) \underbrace{\bar{\psi}(x)}_{\psi(-x)} \, d\psi \quad (\text{i.e., } \hat{\hat{f}} = f(-x)). \quad (dx, d\psi) \text{ is called self-dual.}$$

we've seen that if K = local field, then $K \cong \hat{K}$, w/ $y \mapsto \psi_y$

Q: what Haar measure on K, dx s.t. (dx, dx) is self dual.

e.g. 1. $/\mathbb{R}$. $f(x) = e^{-x^2}$, (μ, μ).

2. $/\mathbb{C}$ $f(z) = e^{-|z|^2}$ (2Lebesgue, 2Lebesgue).

3. $/\mathbb{Q}_p$ $f(x) = \chi_{\mathbb{Z}_p}$. $\hat{f}(\psi_y) = \int_{\mathbb{Z}_p} e^{2\pi i xy} \, dx = 1_{\mathbb{Z}_p}$ so (μ, μ) is self-dual.

$/K$ $f(x) = \chi_{\mathcal{O}_K}$ $\hat{f}(\psi_{L,y}) = \int_{\mathcal{O}_K} e^{2\pi i \, tr(xy)} \, dx = \chi_{\mathcal{C}}$

where $\mathcal{C} = \{x \in K \mid Tr_{K/\mathbb{Q}_p}(xy) \in \mathbb{Z}_p, \ \forall y \in \mathcal{O}_K\}$ ideal of

$D_K = \mathcal{C}^{-1}$ which is the different of K (ramification divisor)

So $\hat{\chi}_{\mathcal{O}_K} = \chi_{\mathcal{C}}$. $\hat{\hat{\chi}}_{\mathcal{O}_K}(0) = \int_{\mathcal{C}} 1 \, dx = (N D_K)^{-1} \cdot \chi_{\mathcal{O}_K}(0)$.

Hence normalized by $(N D_K)^{1/2}$ (multiplied all together on all places will be $\sqrt{|d_K|}$).

~~Tate's~~ Global Fields

Set up. V - set. $\{G_v\}$ LCAG's indexed by V.
$S_\infty \subseteq V$ finite subset $H_v \subseteq G_v$ cpt open subgp $\forall v \notin S_\infty$
μ_v Haar meas. on G_v, $\mu_v(H_v)=1$ $\tilde{\forall} v \notin S_\infty$

Def. $G = \prod'_{v\in V}(G_v, H_v)$ local base at (1) is $\mathcal{B} = \{\prod U_v \mid U_v = H_v, \tilde{\forall} v \in V\}$

Thm (1) G - LCAG (Tychnoff).
(2) $U = \prod U_v \in \mathcal{B}$, then $\mu(U) := \prod_{v\in V} \mu_v(U_v)$ defines a Haar measure μ on G.
(3) $\hat{G} = \prod'_{v\in V}(\hat{G}_v, H_v^{\perp})$

Now $V = V_K$, (K = # field) $S_\infty = V_\infty$ $G_v = K_v$, $H_v = \mathcal{O}_{K_v}$
Then $G = \mathbb{A}_K$, similarly let $G_v = K_v^\times$, $H_v = \mathcal{O}_{K_v}^\times$, then $G = \mathbb{I}_K$.
Recall $\mu_v(\mathcal{O}_{K_v}) = (ND)^{-1/2}$ for v-nonarchimedean
$\mu_{\mathbb{R}} = \text{Leb.}$ $\mu_{\mathbb{C}} = 2\text{Leb.}$

Example $B = \prod\{x_v \in K_v \mid |x_v| \le 1\}$. Then $\mu(B) = \Delta^{-1/2} \cdot 2^r \cdot (2\pi)^s = \dfrac{2^r \cdot (2\pi)^s}{\sqrt{\Delta}}$

Example Let $D = \prod_{v\in V} \mathcal{O}_{K_v} \times F$, F is a fundamental parallelogram for the lattice $\mathcal{O}_K$ in $\mathbb{R}^r \times \mathbb{C}^s$. D is a fund. domain for $\mathbb{A}_K/K$.
$\mu(D) = \frac{1}{\sqrt{|\Delta_K|}} \cdot 2^{-s}\sqrt{|\Delta_K|} \cdot 2^s = 1 \Rightarrow \text{vol}(\mathbb{A}_K/K) = 1.$

Let ψ_v be standard char. on K_v : $\psi_{\mathbb{Q}_p}(x) = e^{-2\pi i x}$ (forget integral part)

$\psi_{\mathbb{R}}(x) = e^{-2\pi i x}$ $\quad \psi_v(x) = \psi_{\mathbb{Q}_p \text{ or } \mathbb{R}} \circ \mathrm{tr}_{K_v/\mathbb{Q}_p \text{ or } \mathbb{R}}$

check that $\psi = \prod_{v \in V_K} \psi_v \in \widehat{\mathbb{A}_K}$ has $\psi(K) = 1$.

$\mathcal{O}_{K_v}^{\perp} = \mathcal{O}_{K_v}$ in $K_v = \widehat{K_v}$ $\forall$ unramified places $\Rightarrow \widehat{\mathbb{A}_K} = \mathbb{A}_K$.

Thm $K^{\perp} = K$ in $\mathbb{A}_K$

pf: $K^{\perp} \overset{\text{def}}{=} \{\chi \in \widehat{\mathbb{A}_K} \mid \chi(\alpha) = 1 \quad \forall \alpha \in K\}$.

$= \{(x_v) \in \mathbb{A}_K \mid \psi(\alpha x) = 1 \quad \forall \alpha \in K\}$.

we note $K^{\perp} \supseteq K$, $K^{\perp} = (\mathbb{A}_K/K)^{\wedge} = (\text{cpt})^{\wedge} = \text{discrete}$

$\left.\begin{array}{l} K^{\perp}/K \subseteq \mathbb{A}_K/K \text{ is discrete in a cpt} \Rightarrow K^{\perp}/K \text{ finite} \\ K^{\perp} \text{ is a } K\text{-linear space} \end{array}\right\} \Rightarrow K^{\perp} = K$

Thm (Tate's RR) $f \in L^1(\mathbb{A}_K)$ nice enough, then $\forall y \in \mathbb{I}_K$,

$$\sum_{\alpha \in K} f(\alpha y) = \frac{1}{\|y\|} \sum_{\alpha \in K} \hat{f}\left(\frac{\alpha}{y}\right)$$

pf sketch: Let $f_y(x) = f(xy)$, reduces to Poisson Summation.

$$\int_H f = \int_{H^{\perp}} \hat{f}.$$

Thm RR: Now Let $K/\mathbb{F}_q$ be a function field. Let $\mathrm{Div}(K) = \{\sum_p n_p P\}$.

$\exists\ \mathcal{K} \in \mathrm{Div}(K)$ s.t. $\forall D \in \mathrm{Div}\, K$. $\ell(D) - \ell(\mathcal{K} - D) = 1 + \deg D - \underset{g(K)}{\underbrace{\ell(\mathcal{K})}}$

pf. Fix $\mathbb{F}_q(T) \subseteq K$ sep., V_K be places, $\forall p$ a place

$$|x|_p = q^{-v_p(x) \deg p}.$$

Def: $\mathbb{I}_K \longrightarrow \mathrm{Div}\, K \qquad q^{-\deg D(y)} = |y|$

$y \longmapsto D_y$

Let $B = \prod_{p\in V_K} \mathcal{O}_{K_p} \subseteq \mathbb{A}_K$ then $q^{\ell(D_y)} = \#\{f\in K \mid f\in y$.

$= \sum_{\alpha\in K} \chi_B(\alpha y)$.

By defn $\widehat{\chi_{\mathcal{O}_{K_p}}}(a) = \int_{\mathcal{O}_{K_p}} \psi_p(ax)\,dx = (ND_p)^{-1/2}\chi_{D_p^{-1}}(a)$.

D_p gen. by $z_p \in \mathcal{O}_{K_p}$. Then $z=(z_p)\in \mathbb{I}_K$, $\mathcal{K} = D_z$

$$\sum_{\alpha\in K}\chi_B(\alpha y) = \frac{1}{\|y\|}\Big(\prod_{p\in V_K}(ND_p)^{-1/2}\Big)\cdot\sum_{\alpha\in K}\boxed{\chi_{z^{-1}B}(\alpha y^{-1})} = \chi_B(\alpha z y^{-1})$$

$$q^{\ell(D_y)} = q^{\deg(D_y)}\, q^{-\frac{\deg\mathcal{K}}{2}}\, q^{\ell(\mathcal{K}-D_y)}$$

$$\ell(D_y)-\ell(K-D_y) = \deg(D_y) - \frac{\deg \mathcal{K}}{2}.$$

Let $D_y = \mathcal{K}$, then $g-1 = \frac{\deg\mathcal{K}}{2}$.

So $\ell(D_y)-\ell(K-D_y) = \deg(D_y) - g+1$.